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Simple solution of velocity profiles of laminar flows in channels of various cross-sections used in field-flow fractionation

Jiří Pazourek* and Josef Chmelík

Institute of Analytical Chemistry, Czechoslovak Academy of Sciences, Veveří 97, 611 42 Brno (Czechoslovakia)

ABSTRACT

A simple method for the two-dimensional description of the flow velocity profile of Newtonian liquids in narrow channels is presented. This procedure is applied to a few types of cross-sections used in field-flow fractionation because the solutions of the flow velocity profiles are necessary for the theoretical description of the separation process in which the flow plays an active role. Limitations of the approach are discussed, and published results on this subject are compared.

INTRODUCTION

Field-flow fractionation (FFF) is an analytical method based on simultaneous actions of physical field forces and the flow of the carrier liquid passing through the separation channel [1]. The carrier liquid takes an active part exhibiting the non-uniform flow velocity profile caused by viscosity effects and by the channel cross-section. Vectors of effective forces of the field and of the carrier liquid are mutually perpendicular: the field acts across the channel and the liquid flows longitudinally. Having different properties and consequently different spatial distributions due to the action of the field, components of a sample migrate along the channel with different elution velocities and thus separation occurs. It follows from the principle of FFF that the most important exploitations are separations of very high-molecular-weight samples; FFF thus holds a promising position in separations of polymers and particles, especially those of biological origin. These compounds, with respect to the properties of biological substances, could be irreversibly changed under conditions currently used in other separation methods, *e.g.*, high pressures, organic solvents and multiple transfers from the mobile to the stationary phases [2].

Channels of various cross-sections can be employed in FFF. Channels of rectangular cross-sections are commonly used for the classical mode of exponential concentration distribution [3] and in some instances for the focusing mode [4–7]. A disadvantage of this cross-section for focusing techniques is the symmetry of the flow velocity profile. This fact implies the application of asymmetric flow profiles in channels of trapezoidal cross-sections for focusing techniques [8–10]. Channels of circular cross-section are used, for example, in FFF techniques operating with an internally induced field (pressure FFF) [11] or with an external electrical field [12].

In all these instances a theoretical description of the separation is impossible without a knowledge of a mathematical expression of the velocity profile of the carrier liquid. While the velocity profiles for the channels of rectangular, circular and elliptical cross-sections have been solved [13–15], the sofar published results for the channels of trapezoidal and parabolic cross-sections [16,17] have not brought generally applicable solutions. This work is concentrated on a simple procedure for finding a two-dimensional description of the flow velocity profiles in channels which are or can be used in FFF.

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THEORY

For a unidirectional, horizontal and steady-state laminar flow of an incompressible viscous liquid in a channel of given length L, the general Navier–Stokes equations hold in the following form (the coordinate system is shown in Fig. 1):

$$\nabla^2 \vec{v} = -\frac{G}{\eta} \tag{1}$$

where ∇^2 is the Laplacean, $\vec{v}$ is the z-component of the velocity vector of the streamline, G is equal to $-\partial p/\partial z$ or approximately to $\Delta p/L$ (Δp is the difference in pressures between the inlet and outlet of the channel) and η is the fluid viscosity.

The solution of eqn. 1 is simple for two infinitely wide parallel planes with a distance w ($\nabla^2 = \partial^2/\partial y^2$):

$$v(y) = \frac{\Delta p}{2\eta L}\left(\tfrac{1}{4}w^2 - y^2\right) \tag{2}$$

When we rewrite eqn. 2 in the form

$$v(x,y) = \frac{\Delta p}{k\eta L}\left[\tfrac{1}{4}w^2(x) - y^2\right] \tag{3}$$

where for the above mentioned case $k = 2$ and x is a real constant, the channel is divided into abstract elements coplanar to the plane yz and eqn. 3 describes the velocity profile in such an arbitrary element x.

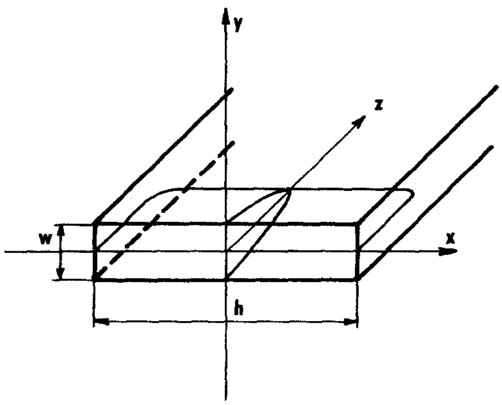

Fig. 1. Channel of rectangular cross-section. The velocity profile is schematically represented by the parabola in the plane yz and by the curve in the plane xz (see eqns. 2 and 5 for $x = 0$ or $y = 0$), h is the channel width and w is the channel height.

For the real flow velocity profile of the liquid in a rectangular cross-section channel with cross-sectional dimensions h and w (see Fig. 1) and for $h \gg w$, Takahashi and Gill [14] derived the relationship

$$v(x,y) = v(y)f(x) \tag{4}$$

where $v(y)$ is the velocity profile defined by eqn. 2 and

$$f(x) = 1 - \frac{\cosh(\sqrt{3}a2x/h)}{\cosh(\sqrt{3}a)} \tag{5}$$

where $a = h/w$. The notation of the cross-section, with h as the larger and w as the smaller dimension, could seem to be unusual for the rectangular cross-section, but in the following cases of triangular and trapezoidal cross-sections, this approach is common. That is why this notation is also used for the rectangular cross-section channel. The course of eqn. 4 is schematically drawn in Fig. 1 as the curve in the plane xz for $y = 0$ and as the parabola in the plane yz for $x = 0$.

Eqn. 4 does not fulfil eqn. 1 because of the term $f(x)$. We used the term $f(x)$ to display approximate flow velocity profiles in real channels. However, for most analytical purposes, the central parts of channels are utilized. With respect to this fact (expressed by the condition $|x| < h/2$), eqn. 5 is approximately equal to 1 and eqn. 4 converts into eqn. 2, which does fulfil eqn. 1.

To solve the problem of the velocity profile in channels with non-rectangular cross-sections, let us start with eqn. 3. If we are able to express $w(x)$ as a function of the variable x that describes the width of the cross-section of the channel in a point x, then we can substitute $w(x)$ for this function and provide a function of two variables: $v(x,y)$. To become the solution of eqn. 1, the new function must satisfy eqn. 1. This appropriate modification can be achieved by substituting $v(x,y)$ into eqn. 1 and the following calculation of k, however, only for polynomial functions $v(x,y)$ of second or first order with respect to both x and y, so that their second derivatives are constant. This limitation is fulfilled by the functions $w(x)$ of the types $w(x) = \sqrt{ex+f}$, $w(x) = \sqrt{ex^2+f}$ and $w(x) = ex+f$, where e and f are constants. Therefore, the above-described procedure will be further applied to these possible cross-sections of the channels.

The triangular cross-section is described by the expression $w(x) = 2q(x + h/2)$, where $q = \tan(\beta/2)$ (the coordinate system is shown in Fig. 2). After substituting this equation for $w(x)$ in eqn. 3 and calculating of k, we obtain

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot \frac{1}{1-q^2} [q^2(x + \tfrac{1}{2}h)^2 - y^2] \quad (6)$$

If we use the equation for the mean velocity:

$$\langle v(y)\rangle_x = 1/h \int_{-h/2}^{h/2} v(x,y)\,\mathrm{d}x, \quad y = \text{constant} \quad (7)$$

we obtain the mean velocity in the plane xz ($y = 0$):

$$\langle v\rangle_x = \frac{\Delta p}{6\eta L} \cdot \frac{q^2}{1-q^2} \cdot h^2 \quad (8)$$

and then we can write eqn. 6 as

$$v(x,0) = 3\,\langle v\rangle_x\,(\tfrac{1}{2} + x/h)^2 \quad (9)$$

which is valid for each $y = cw(x)$, where $c \in \langle 0,1/2\rangle$. The analogous process yields the following results.

For the trapezoidal cross-section $w(x) = 2q(x + K)$ (see Fig. 2) the following holds:

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot \frac{q^2K^2}{1-q^2} [(1 + x/K)^2 - y^2/q^2K^2] \quad (10)$$

$$\langle v\rangle_x = \frac{\Delta p}{2\eta L} \cdot \frac{q^2K^2}{1-q^2} (1 + h^2/12w^2) \quad (11)$$

$$v(x,0) = \frac{\langle v\rangle_x}{(1 + h^2/12K^2)} (1 + x/K)^2 \quad (12)$$

For the parabolic cross-section with the upper restricting wall we can write $w(x) = 2\sqrt{2P(x + h/2)}$ (the coordinate system is shown in Fig. 3) and

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot hP(1 + 2x/h - y^2/hP) \quad (13)$$

$$\langle v\rangle_x = \frac{\Delta p}{2\eta L} \cdot hP \quad (14)$$

$$v(x,0) = \langle v\rangle_x(1 + 2x/h) \quad (15)$$

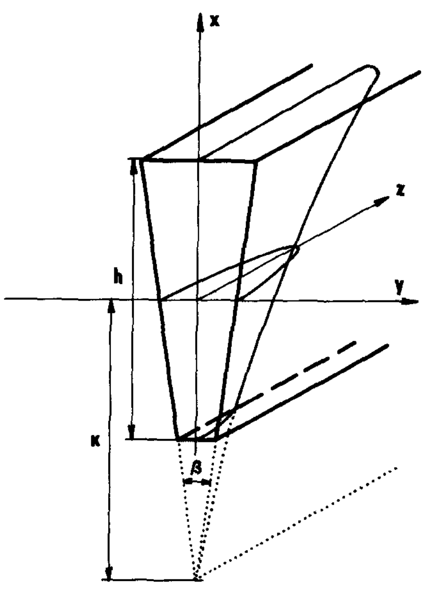

Fig. 2. Channel of trapezoidal cross-section. The velocity profile is schematically represented by the curves in the planes yz or xz (see eqn. 20 for $x = 0$ or $y = 0$), h is the channel height, w is the channel width, β is the angle between the side walls of the channel and K is the distance of the apex line from the axis z. The dotted line represents the channel of triangular cross-section and the corresponding velocity profile.

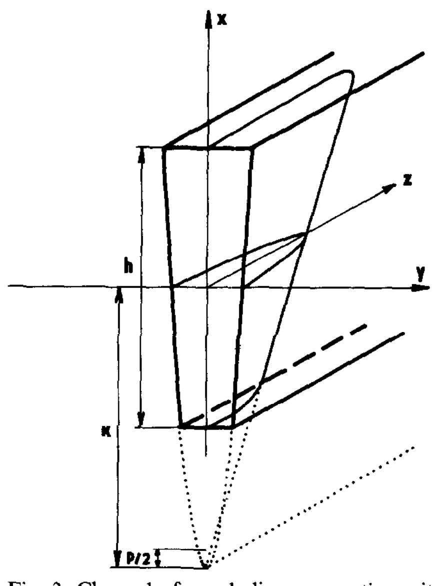

Fig. 3. Channel of parabolic cross-section with two restricting walls. The velocity profile is represented schematically by the curves in the planes yz or xz (see eqn. 21 for $x = 0$ or $y = 0$), h is the height of the channel, P is the parameter of the parabola of the edge of the channel cross-section and K is the distance of the apex line from the axis z. The dotted line represents the channel with only one limiting (upper) wall and the corresponding velocity profile.

For a parabolic cross-section with two restricting walls, the following relationship holds:

$$w(x) = 2\sqrt{2P(x + K)}$$

(see Fig. 3) and, consequently,

$$v(x,y) = \frac{\Delta p}{\eta L} \cdot KP\,(1 + x/K - y^2/2KP) \qquad (16)$$

$$\langle v \rangle_x = \frac{\Delta p}{\eta L} \cdot KP \qquad (17)$$

$$v(x,0) = \langle v \rangle_x (1 + x/K) \qquad (18)$$

It is obvious from the procedure of derivations of the above-mentioned expressions that eqns. 6, 9, 10, 12, 13, 15, 16 and 18 do not generally vanish for $x = \pm h/2$ (the upper and lower walls of the channel). Similarly to the case with the rectangular cross-section channel, we can eliminate this fact by rearranging these equations according to Takahashi and Gill [14]. However, one has to keep in mind that the validity of the equations generated by this procedure is limited only for $|x| < h/2$.

For both the trapezoidal and parabolic cross-sections we can write [provided that $h \gg w(x)$ for $x \in \langle -h/2, h/2 \rangle$]

$$a' = h/w(x) \qquad (19)$$

By multiplying eqns. 10 and 16, respectively, by $f(x)$ from eqn. 5 in which a' is used, the final velocity profile is obtained. Hence, for the trapezoidal cross-section

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot \frac{1}{1 - q^2} [q^2(x + K)^2 - y^2] \cdot \left[1 - \frac{\cosh(\sqrt{3}a' \cdot 2x/h)}{\cosh(\sqrt{3}a')}\right] \qquad (20)$$

and analogously for the parabolic cross-section

$$v(x,y) = \frac{\Delta p}{2\eta L} [2P(x + K) - y^2] \cdot \left[1 - \frac{\cosh(\sqrt{3}a' \cdot 2x/h)}{\cosh(\sqrt{3}a')}\right] \qquad (21)$$

The courses of eqns. 20 and 21 are shown in Fig. 4. In Figs. 2 and 3, the final velocity profiles are schematically drawn for $y = 0$ in the plane xz and for $x = 0$ in the plane yz, respectively.

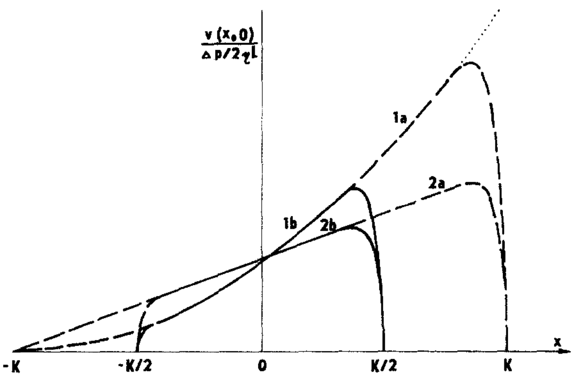

Fig. 4. Velocity profiles in channels of triangular, trapezoidal and parabolic cross-sections. Curves 1a and 1b represent the courses of eqn. 20 and curves 2a and 2b those of eqn. 21 for the case $h = K$ (two limiting walls) (full lines 1b and 2b) and for the case $h = 2K$ (one restricting wall) (dashed line). The dotted line displays the course of eqn. 6. Selected values: $\beta = 5°$; $P = 0.001$; $y = 0$.

In the following two cases, we can obtain well known relationships [15] using the same basic procedure. For a circular cross-section with radius R [$w(x) = 2\sqrt{R^2 - x^2}$, Fig. 5], we provide

$$v(x,y) = \frac{\Delta p}{4\eta L} (R^2 - x^2 - y^2) \qquad (22)$$

and for the elliptic cross-section with semi-axes a,b [$w(x) = 2a\sqrt{1 - x^2/b^2}$, Fig. 6] analogously

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot \frac{a^2 b^2}{a^2 + b^2} (1 - x^2/b^2 - y^2/a^2) \qquad (23)$$

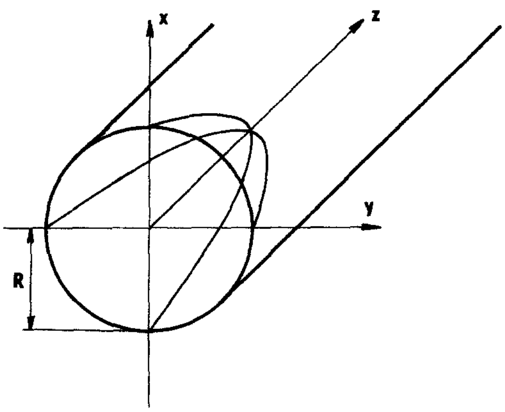

Fig. 5. Channel of circular cross-section. The velocity profile is represented schematically by two parabolas in the planes xz or yz (see eqn. 22 for $y = 0$ or $x = 0$); R is the radius of the tube.

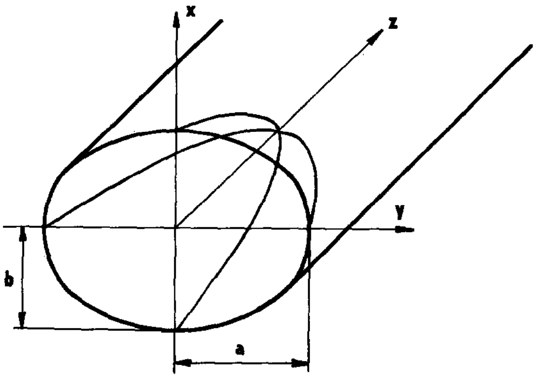

Fig. 6. Channel of elliptical cross-section. The velocity profile is schematically represented by two parabolas in the planes xz or yz (see eqn. 23 for $y = 0$ or $x = 0$); a and b are the semi-axes of the channel cross-section.

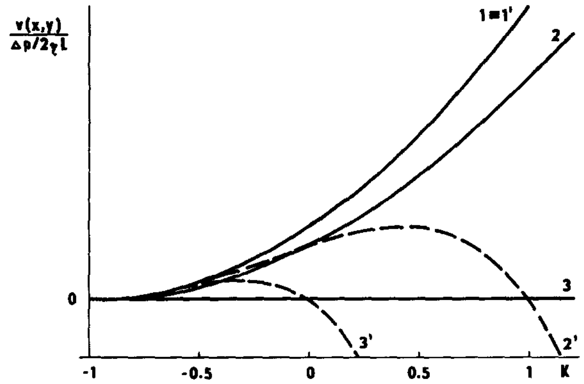

Fig. 7. Comparison of solutions of the velocity profile in a channel of triangular cross-section. Presented are the solution of eqn. 10 (full line) and the solution of ref. 16 (eqn. 24) (dashed line). Values used: $y = 0$ (curves 1 and 1') (plane of channel symmetry), $y = w(x)/4$ (curves 2 and 2') and $y = w(x)/2$ (curves 3 and 3') (the side walls of the channel). Selected values: $K = h/2$; $\beta = 5°$.

DISCUSSION

Several workers have tried to find the mathematical description of the velocity profiles in channels with non-rectangular cross-sections.

Janča and Jahnová [16] solved the cases of channels with trapezoidal and parabolic cross-sections. Their solutions are mathematical modifications of the velocity profile (eqn. 4) according to Takahashi and Gill [14]. The flow velocity profile for the trapezoidal cross-section obtained by Janča and Jahnová [16] can be written as [without the term $f(x)$, see eqn. 5]

$$v(x,y) = \frac{\Delta p}{2\eta L} \cdot q^2K^2(1 - y^2/q^2K^2)(1 + x/K)^2 \quad (24)$$

Fig. 7 shows the courses of eqns. 10 and 24 for $y = cw(x)$ ($c = 0, 0.25, 0.50$). It is obvious that eqn. 24 has a different course for $y < > 0$ and even negative values. These authors' solution for the parabolic cross-section exhibits analogous differences (see Fig. 8):

$$v(x,y) = \frac{\Delta p}{\eta L} \cdot KP\,(1 - y^2/2KP)(1 + x/K) \quad (25)$$

The two solutions have another disadvantage as they do not satisfy the general eqn. 1 [regardless of the term $f(x)$]. This discrepancy was introduced by Wičar [18].

Janča and Chmelík [17] published the following relationship for the description of the flow velocity profile in a channel with a trapezoidal cross-section:

$$v(x) = \frac{\langle v \rangle_x}{1 + \frac{1}{3}\tan^2\beta}\,[1 + (2x/h)\tan\beta]^2 \quad (26)$$

which was derived from eqn. 24 for $y = 0$ and $\beta \to 0$. This equation may be obtained from eqn. 24 or 10 but only for $h/w(0) = 2$, which contradicts the premise conditioning the validity of the solutions of

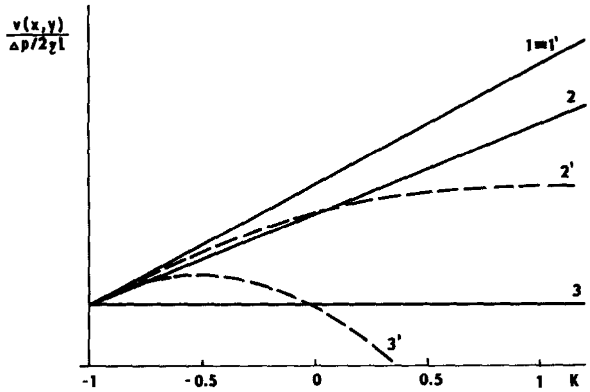

Fig. 8. Comparison of solutions of the velocity profile in a channel of parabolic cross-section with one restricting wall. Presented are the solution of eqn. 16 (full line) and the solution of ref. 16 (eqn. 25) (dashed line). Values used: $y = 0$ (curves 1 and 1') (plane of channel symmetry), $y = w(x)/4$ (curves 2 and 2') and $y = w(x)/2$ (curves 3 and 3') (the side walls of the channel). Selected values: $K = h/2$; $P = 0.001$.

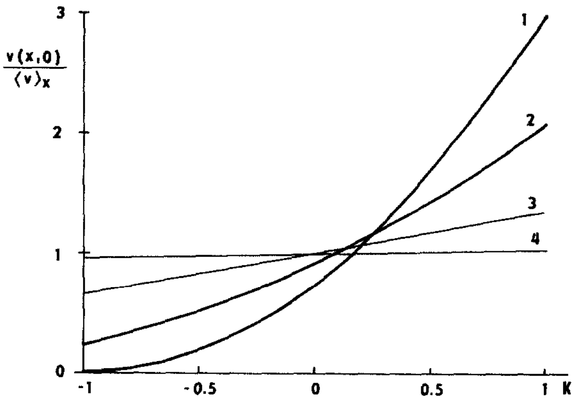

Fig. 9. Comparison of solutions of the velocity profiles in channels of triangular and trapezoidal cross-sections. Presented are solutions of eqn. 9 (curve 1) and eqn. 12 (curve 2). Curves 3 and 4 show the course of the solution of ref. 17 (eqn. 26) for β = 10° and β = 1°, respectively.

eqns. 4 and 24, respectively: $h \gg w$. In Fig. 9 the different course of eqn. 26 to that of eqns. 9 and 12 can be seen.

Wičar [18] solved the case of the flow velocity profile in a channel of a triangular cross-section without the restricting upper wall. His solution is identical with eqn. 6 (see Fig. 4, dotted line).

Eqn. 20 displays the course of the flow velocity profile in the plane xz ($y = 0$) for the trapezoidal cross-section channel. In the central part ($|x| < h/2$) one can see the parabolic shape of the flow velocity profile (see Fig. 2). In the case of a parabolic cross-section channel (eqn. 21) and under the same conditions, the shape of the flow velocity profile is linear (see Fig. 3). The solution of the flow velocity profile for a channel with a circular cross-section is a circular paraboloid (see eqn. 22 and Fig. 5). For the elliptical cross-section channel the solution is an elliptical paraboloid (see eqn. 23 and Fig. 6).

As a comparison criterion for channels of different cross-sections, the limiting ratio defined as

$$R = \lim_{\substack{a' \to \infty \\ x \to h/2}} v_{\max}(x,0)/\langle v \rangle_x$$

can be used [$v_{\max}(x,0)$ is the maximum velocity in the plane xz ($y = 0$) and the other symbols have the above-mentioned meanings]. As follows from eqns. 9 and 15, for channels with triangular cross-sections this limiting ratio is equal to 3 and for parabolic cross-sections to 2. Consequently, channels with triangular or trapezoidal cross-sections are of greater advantage than those of parabolic cross-sections because of their easier design and because of the values of the comparison criterion.

The results of this work permit the design of a theoretical model of the FFF separation process in channels of trapezoidal cross-sections and, consequently, a comparison of the efficiencies of channels with different cross-sections for the individual techniques of FFF.

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